Some general results about the optimal timing of relocation

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Abstract

In this paper we derive general results concerning the optimal switching level in the problem of the optimal relocation policy for a firm that faces two types of uncertainty: one about the moments in which new (and more efficient) sites will become available; and the other regarding the degree of efficiency improvement inherent to each one of these new, yet to be known, potential location places. In particular, we note that the optimal switching level depends on the distribution of the degree of efficiency improvement only through an expected value. Impacts on the final results driven by the characteristics of the firm's original location site, the market environment and the way in which risk is modelled are studied numerically. The overall results are in line with economic intuition.

Keywords: Finance, Re-Location, Project Scheduling, Real Options, Double Poisson Process, Optimal Timing, Erlang Distribution.

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1 Introduction

From 1970 to 1990 the net inflows of foreign direct investment, regarding the GDP percentages, doubled worldwide, as referred by the World Commission on the Social Dimension of Globalization (2004). However, from 1991 to 2000, this process increased significantly, having a growth of almost six fold.

Currently, globalization is the foremost issue on developed economies. Business performance is dramatically stipulated by the international competitiveness, and the awareness of differences regarding location attractiveness has suffered severe changes on the levels of relocation, having increased abruptly.

The relocation of a firm often implies an accurate analysis mainly since it implies a reinvestment. Furthermore, it also appears to bring significant costs to layoff current workers and to call forth major distresses regarding public image. Therefore, risky decision making implies accurate analysis and usually has an irreversible character.

Naturally all the locations are presently known. But there is always the possibility of new developments that can increase drastically the associated efficiency, as for example governmental entities that give incentives to investments in such locations. Therefore relocation is induced by possible uncertainties regarding the way new and more appealing locations turn available and also by the according economic efficiency growth rate. Either one of them has stochastic features, and therefore must be modeled according to a stochastic-economic model of relocation.

For some time now that the academia has shown interest in the economic facilities location. Nonetheless, Sleuwaegen and Pennings (2002) states that the relocation issue hasn't been much explored. Although, interesting statements regarding location literature may be seen in the field of international economics, which has focused on the comparative advantages which tend to determine a location disregarding some other location that is also available. The analysis of effects resultant from singular labor costs and political constraints has been the object of special attention, according to Motta and Thisse (1994); Cordella and Grilo (1998); Collie and Vandenbussche (1999). Some of the possible causes of international ventures is also taken into account in Buckley and Casson (1998); Reuer and Leiblein (2000); Miller and Reuer (1998a,b), still the emphasis being on the outcomes of risk and return rather than on the decision to move from one location to the other. On the contrary, finances have stooped over this subject in a different manner. Studies on capital budgeting have taken into account especially production shifts within the business organization caused by the development of real exchange rates between different economies (see, e.g., De-Meza and van der Ploeg (1987); Capel (1992); Kogut and Kulatilaka (1994a,b); Botteron et al. (2003)). The idea of flexible production units conferring value is present in the studies mentioned previously. Thus, this is seen as an option and as been considered in the real options frameworks.

Nonetheless, and according to our opinion, the studies mentioned previously don't take into account the efficiency's apparent increment required when relocating from one place to the other. The foundations underneath the relocation decision are modeled according to a real options framework, where it's found an optimal behavior which identifies a single change in location.

According to McDonald and Siegel (1986) and Dixit and Pindyck (1994), the decision maker will have to consider, on the one hand, the benefits that will outcome from the increased efficiency afforded by the newly available locations, and, on the other hand, the costs regarding the option of not deferring the relocation. Actually, all possible locations for a business venture are common knowledge. Nonetheless, it is not economically reliable for some industries to settle production units in some locations, due to political, institutional, geographic and economic restraints. Therefore, the concept to be used regarding location will be the one with economic viability. Efficiency will be the prime factor in identifying and distinguishing locations. High levels of efficiency will turn into more outputs with the same level of input (i.e., caused by the workforce's technical competence growth) or to use lower levels of input to generate the same output (i.e., due to lower wages). Either way, the average cost of production will be smaller. We note that in the adoption of new technologies a similar modeling approach was used by Farzin et al. (1998), Huisman (2000) and Huisman and Kort (2003). Information on the availability of new highly efficient locations is modeled according to a Poisson arrival process.

Along the paper we assume a risk-neutral firm, with a constant discount factor, r, like in Dixit and Pindyck (1994), and we let $\pi(.)$ denote the cash flow of the firm. We analyze a dynamic model with an infinite planning horizon, and we assume that when the firm chooses a new location, it incurs a sunk cost investment, that we denote by I, where I can be an arbitrary function. In order to keep notation simple, we use I to denote such relocation cost (but keeping in mind that I may not be constant).

The remainder of the paper is organized as follows. Section 2 presents the model of a firm tackling a relocation decision and facing a stochastic environment, where information about new potentially superior locations is modelled according to a stochastic process. Taking into consideration uncertainties related to (1) the speed at which new, more efficient locations, will become known and (2) the rate of increase in the corresponding efficiency, in Section 3 we derive a general result, stating a kind of robust policy in terms of the distribution of the increase of efficiency. Next, in Section 4 we characterize the optimal relocation time when the jumps in the efficiency follow an Erlang distribution, in Section 5 we provide the corresponding numerical results and the parallel economic rationale, whereas in Section 6 we present the concluding remarks.

A word about notation used in this paper and notably concerning random variables. If X is a random variable, we denote its distribution function by $F_X(.)$ and its density function by $f_X(.)$. Moreover, $\mathbb{E}[X] = \int u dF_X(u)$ denotes its expected value, whereas $\mathbb{E}[X^2] = \int u^2 dF_U(u)$ is its second order moment and $Var[X]$ its variance, with $Var[X] = \mathbb{E}[X^2] - \mathbb{E}^2[X]$. We use the symbol $\square$ to denote end of a proof of a lemma or a theorem. Finally w.p.1. means *with probability one* and i.i.d. means *independent and identically distributed.*

2 Optimal Policy

In this section we develop the model of a risk-neutral firm tackling a relocation decision and facing a stochastic environment, where information about new, potentially superior, locations is modelled according to a stochastic process. The approach to capital budgeting decision making is assumed optimal. Neither the increase in efficiency associated to each new location nor the time that will elapse between the moments in which two sequent efficiency optimizing locations become available are known in advance. Thus we are facing a situation with two levels of uncertainty, the first level corresponding to the time at which a new location becomes available, and the second level to the corresponding increase in the efficiency.

We model the first level of uncertainty using a Poisson process N, with rate λ, that we assume to be independent of the firm. Let $S = \{S_i, i \in I\!N\}$ denote the sequence of events of N. Furthermore, in order to model the uncertainty about the increase of efficiency, we let $\theta(t)$ denote the efficiency of the firm of the best location available at time t, and $\Theta = \{\theta(t), t \in \Re^+\}$ is the corresponding stochastic process, such that for $s \leq t$, $\theta(s) < \theta(t)$ w.p.1 (i.e., we consider only locations that improve, in some degree, the efficiency). Note that Θ is a jump process with strictly increasing jumps. Furthermore, we denote by ψ_0 the efficiency of the firm in its present location, and, without loss of generalization, we assume that $\theta(0) = \psi_0$, so that in fact we are assuming that the firm choosed its initial location in an optimal way.

Furthermore, we let U_i denote the increase in the efficiency at the ith jump of the process Θ:

$$U_i = \theta(S_i) - \theta(S_i^-), \quad i \in I\!N.$$

We assume that $\{U_i, i \in I\!N\}$ is a sequence of positive i.i.d. random variables, identically distributed to the random variable U, and we denote its distribution function by $F_U(.)$. With the previous assumptions, it follows that for any $t \in \Re^+$, $\theta(t)$ can be written as follows:

$$\theta(t) = \psi_0 + \sum_{i=1}^{N(t)} U_i$$

and thus Θ is a compound Poisson process.

It is clear from the description of the problem that the decision to relocate can be stated as a capital budgeting decision problem. Each time a new (and more efficient) location becomes available, the firm has to decide if it stays in the same place (avoiding an investment cost, but loosing the opportunity to produce more efficiently) or if it changes to the new location. Therefore, at each time S_i the firm has to decide between *continuing* in the present location or *stop*, and move to a new location. This decision strongly depends on the relationship between the current efficiency of the firm and the efficiency that it will achieve in the new location. In order to justify a change in location, the corresponding efficiency gains need to overcompensate the resultant relocation costs.

Let $\theta^\star$ denote the value of the efficiency that triggers a relocation, such that if $\theta(t) > \theta^\star$ the firm decides to invest in this new location, wheras if $\theta(t) \leq \theta^\star$ the optimal decision is to stay in its current site and wait for other locations to become available. In addition, we let

$$T^\star = \inf\{t \geq 0 : \theta(t) = \theta^\star\}$$

so that if the firm acts optimally then $T^\star$ is the random variable that denotes the optimal time of relocation.

Following Dixit and Pindyck (1994), we call the value $\theta^\star$ the *optimal switching level*. In this paper we assume the existence and uniqueness of $\theta^\star$.

The optimal switching level can be found using the so-called *value matching condition*, which states that the value of the firm is a continuous function on $\theta^\star$ (see Dixit and Pindyck (1994)), i.e.,

$$F(\theta(T^\star)^-) = F(\theta^\star).$$

where $F(.)$ denotes the value of the firm before its relocation (and including the option of relocation).

If the firm decides to change its current location, then it means that for all $\theta > \theta^\star$

$$F(\theta) = V(\theta) - I \tag{1}$$

where $V(.)$ denotes the value of the firm if it stays in the same location forever, and I is the cost of relocation. On the other hand, if the firm does not change its current location then the value of the firm for $\theta < \theta^\star$ is given by:

$$F(\theta) = \frac{\pi(\psi_0)}{r+\lambda} + \frac{\lambda}{r+\lambda}\left[\int_0^{\theta^\star-\theta} F(\theta+u)dF_U(u) + \int_{\theta^\star-\theta}^{\infty}(V(\theta+u)-I)dF_U(u)\right] \tag{2}$$

where the first component corresponds to the present value of the payoffs inherent to the initial location; the first element in the second component represents the value of the real option to relocate production to a place where a higher level of efficiency will be achieved; and the second element in the second component corresponds to the net present value of the firm after moving to the new location - disbursing the investment cost, I and benefiting from the increase in net cash flows granted by the increased level of efficiency.

3 General Result

In this section we present one of the major contributions of this paper, that extends results presented by Farzin et al. (1998); Huisman and Kort (2003), and Couto (2006).

We note that Equation (2) can be rewritten as follows:

$$\begin{aligned} F(\theta) =& \frac{\pi(\psi_0)}{r+\lambda} + \frac{\lambda}{r+\lambda}\left[\mathbb{E}[F(\theta+U)|U < \theta^\star - \theta]P(U < \theta^\star - \theta)\right. \\ &\left. + \mathbb{E}[V(\theta+U) - I|U \geq \theta^\star - \theta]P(U \geq \theta^\star - \theta)\right]. \end{aligned} \tag{3}$$

If in Equation (3) we set $\theta = \theta^\star$, then we get the following expression for the value of the firm at the optimal switching level:

$$F(\theta^\star) = \frac{\pi(\psi_0)}{r+\lambda} + \frac{\lambda}{r+\lambda}\left(\mathbb{E}[V(\theta^\star + U)] - I\right). \tag{4}$$

Therefore, in view of Equations (1) and (4), we have the following result, which is a major extension of the results previously publisbhed on this subject.

Theorem 3.1 The optimal switching level $\theta^\star$ is the solution of the following equation:

$$(r+\lambda)V(\theta^\star) - \lambda\mathbb{E}[V(\theta^\star + U)] = \pi(\psi_0) + rI. \tag{5}$$

and therefore $\theta^\star$ depends on the distribution of U (the increase in the efficiency of a new location) only through expected values.

The consequence of this result is quite powerful. For instance, if $V(.)$ is a polynomial function of order k, then Theorem (3.1) states that the optimal switching level depends at most on the first k order moments of the random variable U. This result express a certain robustness of the optimal relocation policy in terms of the density distribution of the increments of the efficiency, with major implications in the modelation issue.

We remark in addition that, as $T^\star$ (the optimal time of relocation) is a non-negative random variable, then it follows that $\mathbb{E}[T^\star] = \int_0^\infty (1 - P(T^\star \leq t))\, du$, Ross (1996), where $F_{T^\star}$ denotes the distribution function of $T^\star$, so that:

$$\mathbb{E}[T^\star] = \int_0^\infty (1 - P(T^\star \leq t))\, dt = \int_0^\infty P\left(\theta(t) < \theta^\star\right) dt$$

and therefore:

$$\begin{aligned}
\mathbb{E}[T^\star] &= \int_0^\infty P\left(\theta_0 + \sum_{n=1}^{N(t)} U_n < \theta^\star\right) dt && (6)\\
&= \int_0^\infty \sum_{k=0}^\infty P\left(\sum_{n=1}^{k} U_n < \theta^\star - \theta_0\right) P(N(t)=k)dt\\
&= \int_0^\infty \sum_{k=0}^\infty P\left(\sum_{n=1}^{k} U_n < \theta^\star - \theta_0\right) \frac{e^{-\lambda t}(\lambda t)^k}{k!} dt\\
&= \frac{1}{\lambda} + \int_0^\infty \left[\sum_{k=1}^\infty \left(\int_0^{\theta^\star - \theta_0} f_{\sum_{n=1}^k U_n}(x)dx\right) \frac{e^{-\lambda t}(\lambda t)^k}{k!}\right] dt && (7)
\end{aligned}$$

where $f_{\sum_{n=1}^k U_n}(.)$ denotes the density function of the sum of independent and identically distributed random variables $U_1, U_2, \ldots, U_k$. Using similar arguments, one can prove that:

$$\mathbb{E}[(T^\star)^2] = \frac{2}{\lambda^2} + \int_0^\infty \left[\sum_{k=1}^\infty \left(\int_0^{\theta^\star - \theta_0} f_{\sum_{n=1}^k U_n}(x)dx\right) \frac{e^{-\lambda\sqrt{t}}(\lambda\sqrt{t})^k}{k!}\right] dt. \tag{8}$$

Therefore, in view of Equations (7) and (8), one expect that the properties of the time until relocation may depend significantly on the distribution of the jumps in the efficiency process.

4 Optimal policy in the Erlang case

In this section we assume that the increase in the efficiency is modelled as an Erlang distribution, and we derive the optimal policy. An Erlang distribution with parameters K and μ can be seen as the sum of K independent and identically distributed Exponential random variables, with parameter μ, Ross (2005). We note that the investment decisions are subject to several risk factors eventually independent. Therefore, in modeling this type of problems, makes sense to introduce distribution functions that enable the analytical treatment of more than one state variable, in order to approximate the modeling exercise to corporate reality.

Furthermore, we assume the following Cobb-Douglas production function as a function of the efficiency parameter θ:

$$\pi(\theta) = \phi\theta^2 \tag{9}$$

so that the value of the firm after relocation is

$$V(\theta) = \frac{\phi\theta^2}{r}. \tag{10}$$

Remark that then:

$$\begin{aligned} \mathbb{E}[V(\theta^\star + U)] &= \frac{\phi\mathbb{E}[(\theta^\star + U)^2]}{r} \\ &= \frac{\phi\left((\theta^\star)^2 + \mathbb{E}[U^2] + 2\theta^\star\mathbb{E}[U]\right)}{r} \end{aligned} \tag{11}$$

where $\mathbb{E}[U]$ and $\mathbb{E}[U^2]$ are, respectively, the first and second order moments of the random variable U. Moreover, it follows from Equations (5) and (11) that $\theta^\star$ is the solution of the following equation:

$$\phi r(\theta^\star)^2 - 2\lambda\phi\mathbb{E}[U]\theta^\star - \left(r\pi(\psi_0) + \lambda\phi\mathbb{E}[U^2] + r^2 I\right) = 0. \tag{12}$$

Therefore, if we let

$$a = \phi r, \quad b = \lambda\phi\mathbb{E}[U], \quad \text{and} \quad c = r\pi(\psi_0) + \lambda\phi\mathbb{E}[U^2] + r^2 I$$

then, as a and c are both nonnegative, it follows that $\theta^\star$ is given by:

$$\theta^\star = \frac{b + \sqrt{b^2 + ac}}{a}. \tag{13}$$

As for an Erlang distribution, the two first order moments are given by:

$$\mathbb{E}[U] = \frac{K}{\mu}, \quad \mathbb{E}[U^2] = \frac{K(K+1)}{\mu^2}$$

then

$$\theta^\star = \frac{K\lambda}{r\mu} + \frac{\sqrt{\phi(r^2(\pi(\psi_0) - I\lambda) + \frac{K\lambda(r+Kr+K\lambda)\phi}{\mu^2}}}{r\phi}. \tag{14}$$

In the following theorem we provide results concerning the expected value and variance of the time until relocation.

Theorem 4.1 The expected value and the variance of $T^\star$ are given by:

$$\mathbb{E}[T^\star] = \frac{1}{\lambda} + \frac{1}{\lambda}\sum_{j=1}^{\infty}\left[1 - \frac{\Gamma(jK, (\theta^\star - \theta_0)\mu)}{(jK-1)!}\right] \tag{15}$$

$$Var[T^\star] = \frac{1}{\lambda^2} + \frac{1}{\lambda^2}\sum_{j=1}^{\infty}(2j+1)\left(1 - \frac{\Gamma(jK, (\theta^\star - \theta_0)\mu)}{(jK-1)!}\right) - \frac{1}{\lambda^2}\left(\sum_{j=1}^{\infty}\left[1 - \frac{\Gamma(jK, (\theta^\star - \theta_0)\mu)}{(jK-1)!}\right])\right)^2 \tag{16}$$

where

$$\Gamma(a, b) = \int_b^\infty t^{a-1}e^{-t}dt \tag{17}$$

(the incomplete gamma function).

Proof. Since we assume that $U_i \sim \text{Erlang}(K, \mu)$, it follows that $\sum_{n=1}^{j} U_n \sim \text{Erlang}(jK, \mu)$, as $\{U_i\}$ is a sequence of independent random variables and the Erlang distribution is closed under sums of i.i.d. random variables (Ross (1996)). Therefore in view of Equation (7):

$$\begin{aligned}
\mathbb{E}[T^\star] &= \int_0^\infty \left[e^{-\lambda t} + \sum_{j=1}^{\infty}\int_0^{\theta^\star - \theta_0} \frac{\mu^{jK}x^{jK-1}e^{-\mu x}}{(jK-1)!}dx\frac{e^{-\lambda t}(\lambda t)^j}{j!}\right] dt \\
&= \frac{1}{\lambda} + \int_0^{\theta^\star - \theta_0} e^{-\mu x}\sum_{j=1}^{\infty}\frac{\mu^{jK}x^{jK-1}}{\lambda(jK-1)!}\left[\int_0^\infty \frac{\lambda^{j+1}t^j}{j!}e^{-\lambda t}dt\right] dx \\
&= \frac{1}{\lambda} + \sum_{j=1}^{\infty}\left[\frac{\mu^{jK}}{(jK-1)!}\left(\int_0^{\theta^\star - \theta_0}\frac{e^{-\mu x}x^{jK-1}}{\lambda}dx\right)\right] \\
&= \frac{1}{\lambda} + \sum_{j=1}^{\infty}\left[\frac{\mu^{jK}}{(jK-1)!}\left[\frac{\mu^{-jK}}{\lambda}\left((jK-1)! - \Gamma(jK, (\theta^\star - \theta_0)\mu)\right)\right]\right] \\
&= \frac{1}{\lambda} + \frac{1}{\lambda}\sum_{j=1}^{\infty}\left[1 - \frac{\Gamma(jK, (\theta^\star - \theta_0)\mu)}{(jK-1)!}\right]
\end{aligned}$$

in view of Equation (17).

Similarly, it follows from Equation (8) that

$$
\begin{aligned}
\mathbb{E}[(T^\star)^2] &= \frac{2}{\lambda^2} + \frac{1}{\lambda}\sum_{j=1}^{\infty}\frac{\mu^{jK}}{(jK-1)!}\int_0^{\theta^\star-\theta_0} x^{jK-1}e^{-\mu x}dx \int_0^{\infty}\frac{e^{-\lambda\sqrt{t}}\lambda^{j+1}\sqrt{t}^j}{j!}dt \\
&= \frac{2}{\lambda^2} + \frac{2}{\lambda^2}\sum_{j=1}^{\infty}(j+1)\frac{\mu^{jK}}{(jK-1)!}\int_0^{\theta^\star-\theta_0} x^{jK-1}e^{-\mu x}dx \\
&= \frac{2}{\lambda^2} + \frac{2}{\lambda^2}\sum_{j=1}^{\infty}\frac{(j+1)}{(jK-1)!}\left((jK-1)! - \Gamma(jK,(\theta^\star-\theta_0)\mu)\right) \\
&= \frac{2}{\lambda^2} + \frac{2}{\lambda^2}\sum_{j=1}^{\infty}(j+1)\left(1 - \frac{\Gamma(jK,(\theta^\star-\theta_0)\mu)}{(jK-1)!}\right)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
Var[T^\star] &= \mathbb{E}[(T^\star)^2] - E^2[T^\star] \\
&= \frac{2}{\lambda^2} + \frac{2}{\lambda^2}\sum_{j=1}^{\infty}(j+1)\left(1 - \frac{\Gamma(jK,(\theta^\star-\theta_0)\mu)}{(jK-1)!}\right) - \\
&- \frac{1}{\lambda^2} - \frac{1}{\lambda^2}\left(\sum_{j=1}^{\infty}\left[1 - \frac{\Gamma(jK,(\theta^\star-\theta_0)\mu)}{(jK-1)!}\right])\right)^2 - \frac{1}{\lambda^2}\sum_{j=1}^{\infty}\left[1 - \frac{\Gamma(jK,(\theta^\star-\theta_0)\mu)}{(jK-1)!}\right] \\
&= \frac{1}{\lambda^2} + \frac{1}{\lambda^2}\sum_{j=1}^{\infty}(2j+1)\left(1 - \frac{\Gamma(jK,(\theta^\star-\theta_0)\mu)}{(jK-1)!}\right) - \frac{1}{\lambda^2}\left(\sum_{j=1}^{\infty}\left[1 - \frac{\Gamma(jK,(\theta^\star-\theta_0)\mu)}{(jK-1)!}\right])\right)^2.
\end{aligned}
$$

□

We note, in addition, that Huisman (2000) proved that if $U \sim Exp(\mu)$ then

$$
\mathbb{E}[T^\star] = \frac{(\theta^\star-\theta_0)\mu+1}{\lambda} \tag{18}
$$

$$
\mathbb{E}[(T^\star)^2] = \frac{2+4(\theta^\star-\theta_0)\mu+(\theta^\star-\theta_0)^2\mu^2}{\lambda^2} \tag{19}
$$

whereas Couto (2006) proved that if $U \sim Gamma(2,\mu)$ then

$$
\mathbb{E}[T^\star] = \frac{1}{\lambda} + \frac{\mu}{\lambda}\frac{\theta^\star-\theta_0}{2} - \frac{1}{4\lambda} + \frac{e^{-2\mu(\theta^\star-\theta_0)}}{4\lambda} \tag{20}
$$

$$
\begin{aligned}
\mathbb{E}[(T^\star)^2] = \frac{2}{\lambda^2} + \frac{1}{2\lambda^2}\Bigg(-\frac{5}{4} + e^{-2\mu(\theta^\star-\theta_0)}(\frac{5}{4} - (\theta^\star-\theta_0)\mu) + \\
+\frac{1}{2}(\theta^\star-\theta_0)\mu(6+(\theta^\star-\theta_0)\mu)\Bigg).
\end{aligned} \tag{21}
$$

Using these results in combination with the results of Theorem (4.1), we can prove easily the following relations for the incomplete gamma function.

Corollary 4.2 *The following relations hold for the incomplete gamma function:*

$$\sum_{j=1}^{\infty} \left(1 - \frac{\Gamma(j,a)}{(j-1)!}\right) = a \tag{22}$$

$$\sum_{j=1}^{\infty} j \left(1 - \frac{\Gamma(j,a)}{(j-1)!}\right) = a + \frac{a^2}{2} \tag{23}$$

$$\sum_{j=1}^{\infty} \left(1 - \frac{\Gamma(2j,a)}{(2j-1)!}\right) = \frac{a}{2} + \frac{e^{-2a}-1}{4} \tag{24}$$

$$\sum_{j=1}^{\infty} (j+1) \left(1 - \frac{\Gamma(2j,a)}{(2j-1)!}\right) = \frac{5}{16}(e^{-2a}-1) + \frac{a}{4}(2 + \frac{a}{2}). \tag{25}$$

5 Numerical Illustration

In this section we illustrate some of the results derived in the previous section, using particular instances of the Erlang distribution. In addition, we assume that the production function is as in Equation (9), and that the rate at which new locations become available is $\lambda = 0.5$. According to Theorem (3.1) the optimal switching level depends on the distribution of the jumps U only through the first two moments, $E[U]$ and $E[U^2]$.

We start by assuming specific values for the parameters of interest, namely: output price ($p = 1\ 000$), input price ($w = 250$), discount rate ($r = 0.05$), sunk cost of investiment in a new location ($I = 10\ 000$, here for the sake of the illustration assumed to be constant in time and independent of the relocation site), and initial efficiency ($\theta_0 = 1$). Thus Equation (10) takes the following form:

$$V(\theta) = 20\ 000\theta^2 \tag{26}$$

In the next table we present the optimal switching level for several instances of the shape parameter of the Erlang, K, and the rate parameter, μ. Notice that in these particular cases, the optimal

K	1	2	3	4	5	10
$\mu = 0.1$	2.643	2.628	2.623	2.620	2.619	2.616
$\mu = 10.0$	209.551	207.245	206.465	206.073	205.837	205.364

Table 1: Behaviour of $\theta^\star$ for different instances of the Erlang(K,μ) distribution.

switching level depends more on μ than on K. For $\mu = 0.1$ the first two moments range from 10 and 200 (for $K = 1$) up to 100 and 11000 (for $K = 10$), whereas for $\mu = 10$ they range from 0.1 and 0.2 (for $K = 1$) up to 1 and 1.1 (for $K = 10$). In addition, for the case $\mu = 10$ the first two moments are of the same order, whereas for $\mu = 0.1$ the second moment is at least one order larger than the first moment.

Finally, in order to check the influence of the second moment, we present values regarding the optimal switching level for instances of the Erlang distribution with the same expected value (but different second order moment), which is in accordance with the result of Theorem (3.1). Thus differences on $\theta^\star$ are to be expected as the value of the firm is a quadratic form, and therefore $\theta^\star$ depends both on the first and second order moment.

K	1	2	3	4	5	10
μ	0.1	0.2	0.3	0.4	0.5	1
$\theta^\star$	2.643	4.470	6.421	8.416	10.431	20.607
μ	10	20	30	40	50	100
$\theta^\star$	209.551	414.480	619.377	824.266	1,029.150	2,053.570

Table 2: Behaviour of $\theta^\star$ for different instances of the Erlang(K,μ) distribution, with constant expected value.

Next we present numerical results concerning the optimal switching level when we impose changes in certain parameter values. In particular, we analise the effect on the optimal switching level, $\theta^\star$, of the following parameters: output price p, input price w, investment cost I, discount rate r, and initial efficiency θ_0. We consider three instances of the Erlang distribution with parameters K and $\mu = 0.1$, with $K = 1, 3$ and 5. See Figs. (1-5).

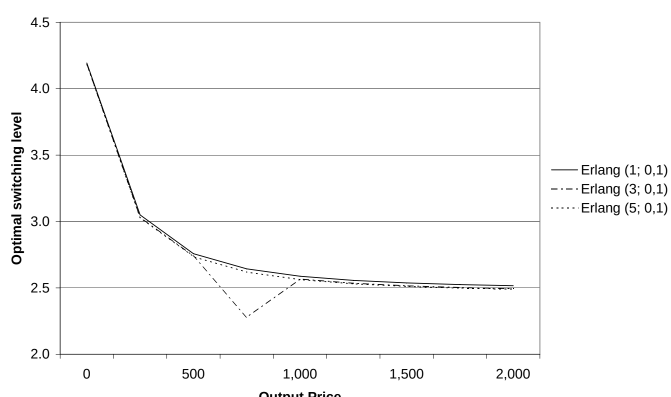

Figure 1: Behavior of the optimal switching level as a function of the output price.

Higher levels of output prices lead naturally to decreases in the value of the option to delay an investment. Therefore, increases in output prices lead to decreases in optimal switching levels (see Fig. (1)). In effect, the level of efficiency that triggers a change in location needs to increase in order to compensate the gross margin reduction, induced by the decreases in output prices.

In the opposite, higher levels of input related costs lead naturally to increases in the value of the option to delay an investment. Therefore, increases in input prices lead to increases in optimal switching levels (see Fig. (2)). In result, the level of efficiency that triggers a change in location needs to increase in order to compensate too the gross margin reduction, induced by the increase in input prices.

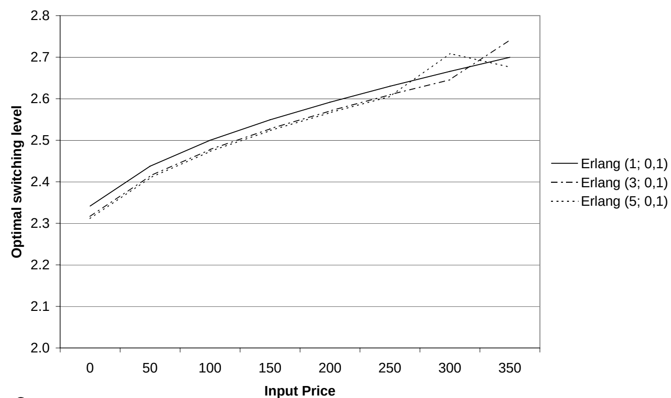

Figure 2: Behavior of the optimal switching level as a function of the input price.

The relationship between optimal switching levels and investment costs follows a similar pattern (see Fig. (3)), and seems to be essentially an increasing linear function of the level of the expected increase in investment costs. Increases in investment costs need to be properly compensated by efficiency increases in order to justify changes in location.

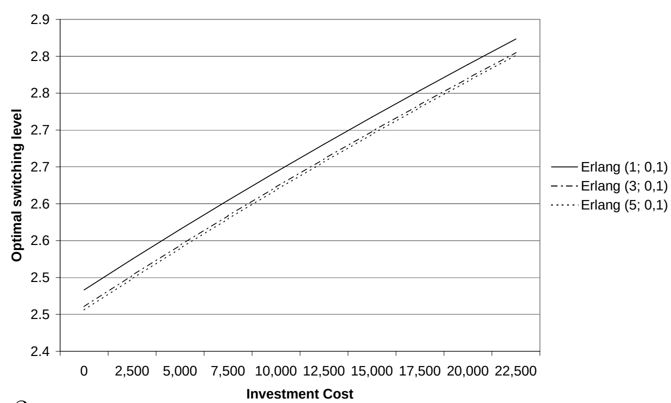

Figure 3: Behavior of the optimal switching level as a function of the investment.

We also present a plot concerning the behaviour of the optimal switching level as a function of the discount rate (see Fig. (4)). Reduced discount rate levels mean smaller time value of money and consequently, a small potential loss for postponing the decision to relocate. In contrast, very high discount rate levels imply untenable delaying costs. The convex shape is in accordance with the economic rationale related to the valuation of all interest rate products.

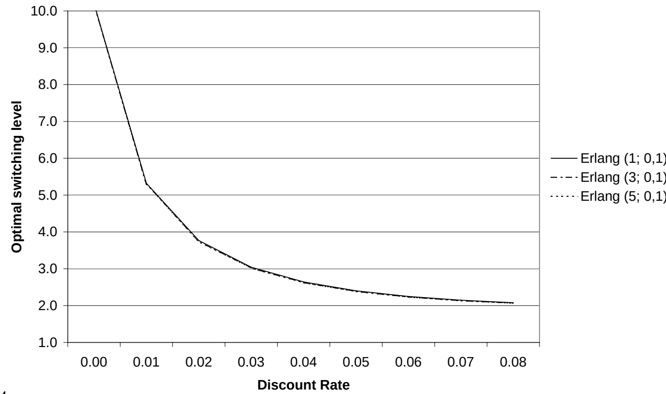

Figure 4: Behavior of the optimal switching level as a function of the discount factor.

Finally we present a plot concerning the behaviour of the optimal switching level as a function of the initial efficiency. Higher levels of initial theta lead naturally to increases in the value of the option to delay an investment. Therefore, increases in initial values of the actual efficiency lead to increases in optimal switching levels (see Fig. (5)). In effect, the level of efficiency that triggers a change in location needs to increase in order to compensate the loss of actual efficiency in the current location.

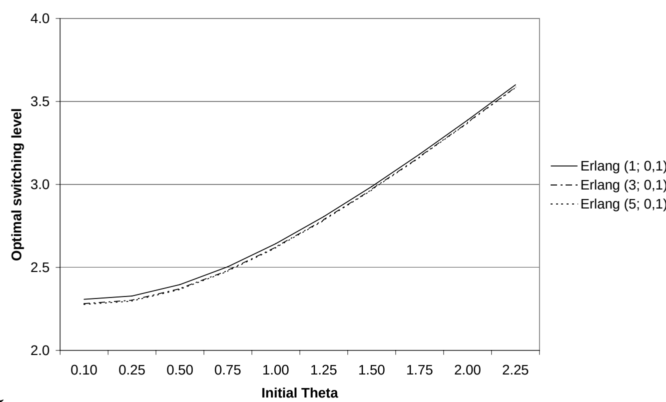

Figure 5: Behavior of the optimal switching level as a function of the initial efficiency.

6 Concluding remarks

The problem of relocation is especially relevant, given its socio-economic implications, in the present period of globalization. The optimal timing of relocation is significantly affected by the uncertainties related to both the expected rhythm that characterizes the arrival of information regarding the availability of new, more efficient location sites, and the degree in efficiency improvement from one location to the other. The focus of the article is on the perceived increase in efficiency needed to justify the decision to relocate from one place to another. Using a stochastic framework, we have analyzed the problem of the optimal timing for the relocation of a firm to move its production site. Given the specific characteristics of the relocation decision, and unlikely any other work that we are aware of in this field, we note that the optimal switching level depends on the distribution of U (the increase in the efficiency of a new location) only through an expected value. Finally numerical outcomes suggest that the results of our model are in accordance with economic rationale.

References

Botteron, P., M. Chesney, and R. Gibson-Asner (2003). Analyzing firms' strategic investment decisions in a real options' framework. *Journal of International Financial Markets, Institutions & Money 13*, 451–479.

Buckley, P. and M. Casson (1998). Models of the multinational enterprise. *Journal of International Business Studies 29*(1), 21–44.

Capel, J. (1992). How to service a foreign market under uncertainty: a real option approach. *European Journal of Political Economy 8*, 455–5475.

Collie, D. and H. Vandenbussche (1999). *Trade, FDI, and Unions*. Working Paper, Cardiff Business School.

Cordella, T. and I. Grilo (1998). Globalization and relocation in a vertically differentiated industry. *CEPR discussion paper*.

Couto, G. (2006). *Opções Reais e Decisão sob Incerteza no Processo de Relocalização*. Ph. D. thesis, Instituto Superior de Economia e Gestão, Technical University of Lisbon, Portugal.

De-Meza, D. and F. van der Ploeg (1987). Production flexibility as a motive for multinationality. *Journal of Industrial Economy 35*, 343–352.

Dixit, A. and R. Pindyck (1994). *Investment Under Uncertainty*. Princeton University Press.

Farzin, Y., K. Huisman, and P. Kort (1998). Optimal timing of technology adoption. *Journal of Economic Dynamics and Control 22*, 779–799.

Huisman, K. (2000). *Technology Investment: a game theoretic real options approach*. Ph. D. thesis, Tilburg University, Department of Econometrics, CentER Dissertation Series, Centre for Quantitative Methods in Eindhoven, Tilburg, The Netherlands.

Huisman, K. and P. Kort (2003). Strategic investment in technological innovations. *European Journal of Operational Research 144*(1), 209–233.

Kogut, B. and N. Kulatilaka (1994a). Operating flexibility, global manufacturing, and the option value of a multi-natial network. *Management Science 40*(1), 123–139.

Kogut, B. and N. Kulatilaka (1994b). Options thinking and platform investments: Investing in opportunity. *California Management Review 36*(2), 52–71.

McDonald, R. and D. Siegel (1986). The value of waiting to invest. *The Quarterly Journal of Economics 101*(4), 707–728.

Miller, K. and J. Reuer (1998a). Asymmetric corporate exposures to foreign exchange rate changes. *Strategic Management Journal 19*, 1183–1191.

Miller, K. and J. Reuer (1998b). Firm strategy and economic exposure to foreign exchange rate movements. *Journal of International Business Studies 29*(3), 493–514.

Motta, M. and J.-F. Thisse (1994). Does environmental dumping lead to delocation? *Journal of Financial and Quantitative Analysis 38*, 563–576.

Reuer, J. and M. Leiblein (2000). Downside risk implications of multinationality and international joint ventures. *Academy of Management Journal 43*(2), 203–214.

Ross, S. (1996). *Stochastic Processes.* New York: Wiley.

Ross, S. (2005). *A First Course in Probability.* Prentice Hall.

Sleuwaegen, L. and E. Pennings (2002). *New Empirical Evidence on the International Relocation of Production.* Working Paper, Erasmus University Rotterdam.